

AN ASYMPTOTIC VERSION OF DUMNICKI'S ALGORITHM FOR LINEAR SYSTEMS IN \mathbb{CP}^2

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ABSTRACT. Using Dumnicki's approach to showing non-specialty of linear systems consisting of plane curves with prescribed multiplicities in sufficiently general points on \mathbb{P}^2 we develop an asymptotic method to determine lower bounds for Seshadri constants of general points on \mathbb{P}^2 . With this method we prove the lower bound $\frac{4}{13}$ for 10 general points on \mathbb{P}^2 .

0. INTRODUCTION

A celebrated conjecture of Nagata [Nag59] predicts that every curve in $\mathbb{P}^2 = \mathbb{CP}^2$ going through $r > 9$ very general points with multiplicity at least m has degree $d \geq \sqrt{rm}$. Cast in the language of Seshadri constants, Nagata claimed in effect that

$$H - \sqrt{\frac{1}{r}} \sum_{j=1}^r E_j$$

is a nef divisor on $\tilde{X} = \text{Bl}_r(\mathbb{P}^2)$, the blowup of \mathbb{P}^2 in the r points, where H is the pullback of a line in \mathbb{P}^2 and E_j are the exceptional divisors over the blown up points.

It is well known that Nagata's conjecture is implied by another conjecture of Harbourne and Hirschowitz about spaces $\mathcal{L}_d(m^r)$ of plane curves of given degree d and multiplicity at least m at r general points [Mir99, CM01]. This conjecture tries to detect those of the spaces $\mathcal{L}_d(m^r)$ which do not have the expected dimension

$$\max\left(-1, \frac{d(d+3)}{2} - r \cdot \frac{m(m+1)}{2}\right).$$

In [Eck05] the author showed that it is not necessary to know all cases of the Harbourne-Hirschowitz conjecture in order to prove Nagata's conjecture:

Theorem 0.1 ([Eck05], Thm.5.1). *Let $r > 9$ be an integer and (d_i, m_i) a sequence of pairs of positive integers such that $\frac{d_i^2}{m_i^2 \cdot r} \xrightarrow{i \rightarrow \infty} \frac{1}{a^2} \geq 1$ and the space $\mathcal{L}_{d_i}((m_i+1)^r)$ has expected dimension ≥ 0 . Then*

$$H - a \cdot \sqrt{\frac{1}{r}} \sum_{j=1}^r E_j$$

is nef on \tilde{X} . In particular, Nagata's conjecture is true for r general points in \mathbb{P}^2 , if $a = 1$.

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In this paper we want to use Dumnicki's Reduction Algorithm [DJ05, Dum06] to prove the non-specialty of linear systems $\mathcal{L}_d(m^r)$, as needed in the theorem. Dumnicki's new idea was to consider linear systems of curves not only going through certain points with at least certain multiplicities, but also the curve equation should contain only monomials from a certain subset of all monomials of degree $\leq d$. He was able to give non-specialty criteria for such linear systems, including the following:

Proposition 0.2 (Dumnicki's non-specialty criterion). *Let $m \in \mathbb{N}$ and let $D \subset \mathbb{N}^2$ such that $\#D = \binom{m+1}{2}$. Consider the linear system $L = \mathcal{L}_D(m)$ of those curve equations $\sum_{(\alpha,\beta) \in D} c_{\alpha,\beta} x^\alpha y^\beta$, $c_{\alpha,\beta} \in \mathbb{C}$, which pass through a given point with multiplicity at least m . Then L is non-special if and only if the points in D do not lie on a curve of degree $m-1$ in \mathbb{R}^2 . In particular, L is non-special if there are m parallel lines l_1, \dots, l_m containing $1, \dots, m$ points in D .*

Proof. See [Dum06, Prop.12]. The last statement follows from Bézout's Theorem. \square

Furthermore, Dumnicki devised a recursive procedure showing the non-specialty of linear systems $\mathcal{L}_D(m_1, \dots, m_r)$ if it terminates in the correct way:

Theorem 0.3 (Dumnicki's reduction algorithm). *Let $m_1, \dots, m_{p-1}, m_p \in \mathbb{N}^*$, let $D \subset \mathbb{N}^2$, and let*

$$F : \mathbb{R}^2 \ni (a_1, a_2) \mapsto r_0 + r_1 a_1 + r_2 a_2 \in \mathbb{R}, \quad r_0, r_1, r_2 \in \mathbb{R},$$

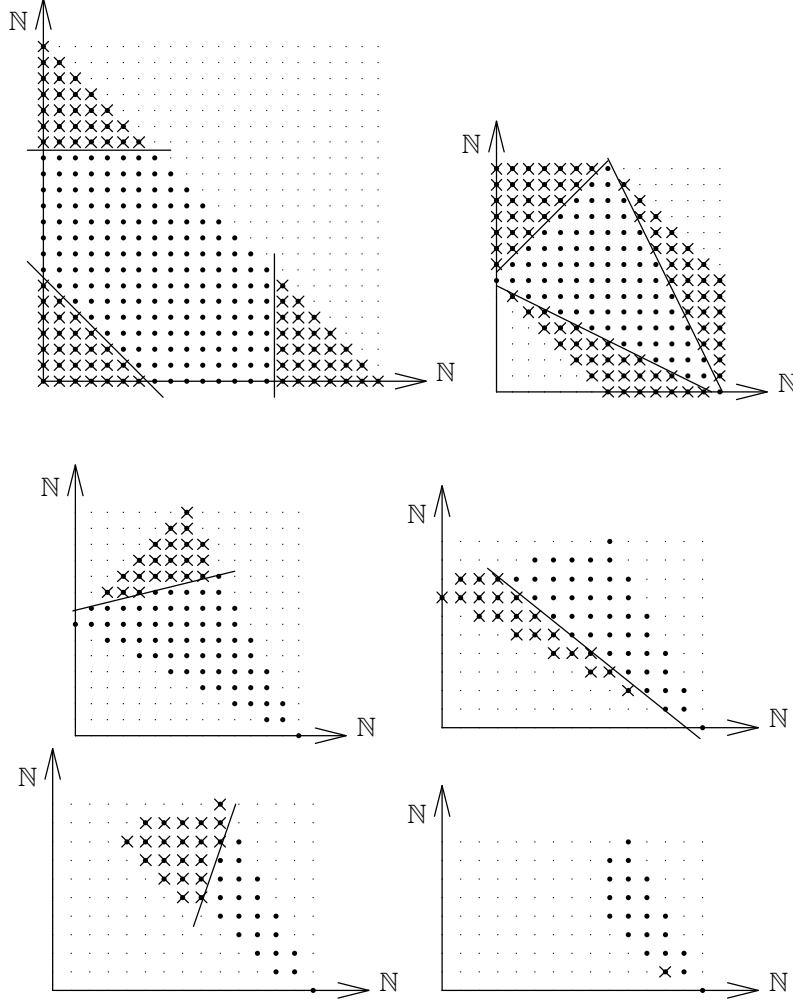
be an affine function. Let

$$\begin{aligned} D_1 &:= \{(a_1, a_2) \in D \mid F(a_1, a_2) < 0\}, \\ D_2 &:= \{(a_1, a_2) \in D \mid F(a_1, a_2) > 0\}. \end{aligned}$$

If $D_1 \cup D_2 = D$ and $L_1 := \mathcal{L}_D(m_1, \dots, m_{p-1})$ is non-special of dimension ≥ 0 , $L_2 := \mathcal{L}_D(m_p)$ is non-special of dimension -1 , then $\mathcal{L}_D(m_1, \dots, m_p)$ is non-special of dimension ≥ 0 .

Proof. See [Dum06, Thm.13]. \square

Dumnicki used this procedure to show the Harbourne-Hirschowitz conjecture up to $m = 42$, but the power and simplicity of it can best be seen by some easy graphical proofs of non-specialty. For example, Dumnicki [Dum06, Ex.37] used a computer to find the following proof for non-specialty of the system $L = \mathcal{L}_{21}(7^{\times 6}, 6^{\times 4}, 1)$:



Since the Nagata conjecture for square-free integers $r > 9$ involves irrational square roots, it seems appropriate to look for an asymptotic version of Dumnicki's reduction algorithm. To this purpose we introduce the following notion:

Definition 0.4. Let $m_1, \dots, m_r \in \mathbb{R}_{>0}$. A subset $P \subset \mathbb{R}_{\geq 0}^2$ contains asymptotically (m_1, \dots, m_r) -non-special systems (of dimension $\geq d$) iff for all $\delta > 0$ and all $k \gg 0$ there exist $m_1^{(k)}, \dots, m_r^{(k)} \in \mathbb{N}$ and a $D_k \subset k \cdot P \cap \mathbb{N}_{\geq 0}^2$ such that

- (i) $\mathcal{L}_{D_k}(m_1^{(k)}, \dots, m_r^{(k)})$ is non-special (of dimension $\geq d$) and
- (ii) $\left| \frac{m_i^{(k)} - km_i}{km_i} \right| < \delta, i = 1, \dots, r.$

With this notion we prove the following method of obtaining bounds on Seshadri constants on \mathbb{P}^2 (see Section 3 for the proof):

Proposition 0.5. *If the set $P := \{x + y \leq 1\} \cap \mathbb{R}_{\geq 0}^2$ contains asymptotically (m^r) -non-special systems of dimension ≥ 0 then*

$$H - m \sum_{j=1}^r E_j$$

is nef on X , where X is the blow-up of \mathbb{P}^2 in r very general points, H is the pullback of a line in \mathbb{P}^2 to X , and the E_i are the exceptional divisors on X .

To show the existence of (m^r) -non-special systems we develop an asymptotic version of Dumnicki's reduction algorithm (see Thm. 2.1), and together with a criterion for asymptotic (m) -non-specialty (see Thm. 2.2), we are able to give the following bound on the Seshadri constant of 10 very general points on \mathbb{P}^2 (see again Section 3 for the proof):

Theorem 0.6. *Let X be the blow-up of \mathbb{P}^2 in 10 very general points, let E_1, \dots, E_{10} be the exceptional divisors on X , and let H be the pull back of a line in \mathbb{P}^2 . Then the divisor*

$$H - \frac{4}{13} \sum_{i=1}^{10} E_i$$

is nef on X .

In recent years many authors tried to give lower bounds for the Seshadri constants of a fixed number of general points on algebraic surfaces, and especially on \mathbb{P}^2 [Xu94, STG02, Har03, HR04, HR05]. Some of these bounds are even better than $\frac{4}{13} \approx 0.307$, which is still not really close to $\frac{1}{\sqrt{10}} \approx 0.3162$: Tutaj-Gasińska [TG03] achieved $\frac{2}{11}\sqrt{3} \approx 0.314$, Biran [Bir99] $\frac{6}{19} \approx 0.3158$, and Harbourne-Roé [HR03] even $\frac{177}{560} \approx 0.31607$. At least, our bound is better than what can be achieved by using the non-specialty of all non-empty linear systems $\mathcal{L}_d(m^{10})$ up to $m \leq 42$, shown by Dumnicki [Dum06]: The expected dimension of $\mathcal{L}_d(41^{10})$ is ≥ 0 iff $d \geq 132$, hence Thm. 0.1 applied to the constant sequence $(132, 40)$ gives the bound $\frac{40}{132} \approx 0.303$. For all other multiplicities $m \leq 42$ the bound gets smaller.

In any case the true interest in this bound lies in the fact that it was shown with an asymptotic method.

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1. MONOTONE REORDERING

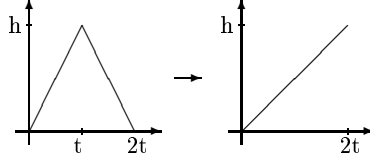
In this section we collect elementary, but useful facts about the *monotone reordering* of functions:

Definition 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a measurable function on a closed interval $[a, b] \subset \mathbb{R}$. Then the monotone reordering $f^\# : (0, b - a] \rightarrow \mathbb{R}$ of f is defined by*

$$t \mapsto \inf \{s : t \leq \text{length of } \{t' \in [a, b] : f(t') \leq s\}\}.$$

This notion will be used to state the criterion of (m) -non-specialty (see Thm. 2.2).

Remark 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a step function. Then $f^\# : (0, b - a] \rightarrow \mathbb{R}$ reorders its steps such that they increase monotonely. Another example is given in the following diagram which shows the monotone reordering of a piecewise-linear function:



Proposition 1.3. *The monotone reordering $f^\#$ of a function $f : [a, b] \rightarrow \mathbb{R}$ is monotonely increasing and lower semi-continuous.*

Proof. If $t_1 < t_2$ then $t_2 \leq \text{length of } \{f(t') \leq s\}$ implies $t_1 \leq \text{length of } \{f(t') \leq s\}$, hence

$$\begin{aligned} f^\#(t_1) &= \inf \{s : t_1 \leq \text{length of } \{f(t') \leq s\}\} \\ &\leq \inf \{s : t_2 \leq \text{length of } \{f(t') \leq s\}\} = f^\#(t_2). \end{aligned}$$

Furthermore, set $s := f^\#(t)$ and assume for given $\epsilon > 0$ that

$$\tilde{t} \leq \text{length of } \{f(t') \leq s - \epsilon\}$$

for all $\tilde{t} < t$. This implies $t \leq \text{length of } \{f(t') \leq s - \epsilon\}$, a contradiction to $f^\#(t) = s$. Hence there exists a $\bar{t} < t$ such that $f^\#(\bar{t}, b - a) > s - \epsilon$, and $f^\#$ is lower semi-continuous. \square

Proposition 1.4. *Let $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ be two measurable functions such that $f_1 \leq f_2$. Then $f_1^\# \leq f_2^\#$.*

Proof. If $f_1 \leq f_2$ then for fixed s ,

$$\text{length of } \{f_1 \leq s\} \geq \text{length of } \{f_2 \leq s\}.$$

This implies for fixed t that

$$\{s : t \leq \text{length of } \{f_1 \leq s\}\} \supset \{s : t \leq \text{length of } \{f_2 \leq s\}\},$$

hence $f_1^\#(t) \leq f_2^\#(t)$. \square

Proposition 1.5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then the monotone reordering $f^\# : (0, b - a] \rightarrow \mathbb{R}$ is also continuous.*

Proof. We already know from Prop. 1.3 that $f^\#$ is lower semi-continuous. Let $t \in (0, b - a]$ and set $s := f^\#(t)$. For $t = b - a$ or $s = \max\{f(t') : t' \in [a, b]\}$ nothing is to prove. Since f is continuous the set $\{t' : s < f(t') < s + \epsilon\}$ is open and non-empty for all $\epsilon > 0$ and has consequently a positive length δ_ϵ . Then

$$\text{length of } \{f \leq s + \epsilon\} \geq \text{length of } \{f \leq s\} + \text{length of } \{s < f < s + \epsilon\} \geq t + \delta_\epsilon,$$

hence $f^\#(t') < s + 2\epsilon$ for all $t' < t + \delta_\epsilon$, and $f^\#$ is upper semi-continuous in t . \square

Theorem 1.6. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on the closed interval $[a, b] \subset \mathbb{R}$. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all closed intervals $[a', b'] \subset [a, b]$ with $(b - a) - (b' - a') < \delta$ the monotone reorderings $f^\# : (0, b - a] \rightarrow \mathbb{R}$ and $(f|_{[a', b']})^\# : (0, b' - a'] \rightarrow \mathbb{R}$ satisfy*

$$\|(f^\#)|_{(0, b' - a']} - (f|_{[a', b']})^\#\|_{\max} < \epsilon.$$

Proof. Since f is continuous on the compact interval $[a, b]$ the function is uniformly continuous on $[a, b]$. Consequently, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|t - t'| < \delta$ implies $|f(t) - f(t')| < \epsilon$.

Claim 1. $|t - t'| < \delta$ also implies $|f^\#(t) - f^\#(t')| < 2\epsilon$.

Proof. Since f is continuous f has a minimum s_{\min} on $[a, b]$ which also must be a lower bound for $f^\#$ on $(0, b - a]$. By construction, length of $\{f \leq s_{\min} + \epsilon\} \geq \delta$, hence

$$|f^\#(t) - f^\#(t')| \leq \epsilon < 2\epsilon \text{ for all } 0 < t \leq t' < \delta.$$

Now let $t' \geq \delta$. If $f^\#(t') - 2\epsilon < s_{\min}$, we have

$$|f^\#(t') - f^\#(t)| \leq f^\#(t') - s_{\min} < 2\epsilon \text{ for } t < t',$$

since $f^\#$ is monotonely increasing by Prop. 1.3.

Otherwise $f^\#(t') - 2\epsilon \geq s_{\min}$, and we show two claims:

Claim 1.1. $f : [a, b] \rightarrow \mathbb{R}$ continuous \Rightarrow length of $\{f < f^\#(t')\} \leq t'$.

Proof. The characteristic function of the sets $\{s - \epsilon < f < s\}$ tend pointwise to 0 for $\epsilon \rightarrow 0$, and they are dominated by the integrable characteristic function of $[a, b]$. Hence, by Lebesgue's dominated convergence,

$$\text{length of } \{s - \epsilon < f \leq s\} \rightarrow 0, \text{ for } \epsilon \rightarrow 0.$$

If length of $\{f < f^\#(t')\} > t'$, this limit would imply the existence of an $\epsilon > 0$ such that

$$\text{length of } \{f < f^\#(t') - \epsilon\} > t',$$

a contradiction to the definition of $f^\#(t')$. \square

Claim 1.2. Length of $\{f^\#(t') - 2\epsilon < f < f^\#(t')\} \geq \delta$.

Proof. $f^\#(t')$ is a value of f on $[a, b]$: Otherwise $f^\#(t')$ would be bigger than the maximum s_{\max} of f on $[a, b]$, hence

$$\text{length of } \{f \leq f^\#(t')\} = \text{length of } \{f \leq s_{\max}\},$$

contradicting the definition of $f^\#(t')$. Furthermore, $f^\#(t') - 2\epsilon > s_{\min}$, hence by continuity, there exists a \bar{t} such that $f(\bar{t}) = f^\#(t') - \epsilon$. But then the construction of δ shows that

$$f((\bar{t} - \delta, \bar{t} + \delta)) \subset (f^\#(t') - 2\epsilon, f^\#(t')).$$

Since w.l.o.g. we can assume that $\delta < b - a$, the interval $(\bar{t} - \delta, \bar{t} + \delta) \cap [a, b]$ has length $\geq \delta$, hence the claim. \square

From these two claims we deduce

$$\begin{aligned} \text{length of } \{f \leq f^\#(t') - 2\epsilon\} &= \\ &= \text{length of } \{f < f^\#(t')\} - \text{length of } \{f^\#(t') - 2\epsilon < f < f^\#(t')\} \\ &\leq t' - \delta, \end{aligned}$$

hence we have $f^\#(t') \geq f^\#(t) > f^\#(t') - 2\epsilon$ for all $t' - \delta < t \leq t'$. This proves Claim 1. \square

Now choose $a', b' \in [a, b]$ such that $d := (b - a) - (b' - a') < \delta$. Since f is continuous it has a maximum M and a minimum m on the compact set $[a', b']$. Define

$$\underline{f}(t) := \begin{cases} m - \epsilon & \text{for } t \in [a, a') \cup (b', b] \\ f(t) & \text{else} \end{cases}, \quad \bar{f}(t) := \begin{cases} M + \epsilon & \text{for } t \in [a, a') \cup (b', b] \\ f(t) & \text{else} \end{cases}$$

Then $\underline{f} \leq f \leq \overline{f}$, hence $\underline{f}^\# \leq f^\# \leq \overline{f}^\#$ by Prop. 1.4, and furthermore

$$\underline{f}^\#(t) = \begin{cases} m - \epsilon & \text{for } t \leq d \\ (f_{|[a', b']})^\#(t - d) & \text{else} \end{cases}, \quad \overline{f}^\#(t) = \begin{cases} (f_{|[a', b']})^\#(t) & \text{for } t \leq b' - a' \\ M + \epsilon & \text{else.} \end{cases}$$

Consequently, for $t \leq b' - a'$,

$$\begin{aligned} |(f_{|[a', b']})^\#(t) - f^\#(t)| &\leq |\overline{f}^\#(t) - \underline{f}^\#(t)| \\ &= \begin{cases} (f_{|[a', b']})^\#(t) - (m - \epsilon) & \text{for } t \leq d \\ (f_{|[a', b']})^\#(t) - (f_{|[a', b']})^\#(t - d) & \text{else.} \end{cases} \\ &< \begin{cases} 3\epsilon \\ 2\epsilon \end{cases} \end{aligned}$$

using $d < \delta$, $m \leq (f_{|[a', b']})^\#(t)$ for all $t \in (0, b' - a']$ and Claim 1 applied to $f_{|[a', b']}$. \square

Proposition 1.7. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous function on $[a, b]$. Then for all $\epsilon > 0$,*

$$\|f - g\|_{\max} < \epsilon \implies \|f^\# - g^\#\|_{\max} < 2\epsilon.$$

Proof. $\|f - g\|_{\max} < \epsilon$ implies $f - \epsilon < g < f + \epsilon$. Since $(f \pm \epsilon)^\# = f^\# \pm \epsilon$, we obtain from Prop. 1.4

$$f^\# - \epsilon \leq g^\# \leq f^\# + \epsilon,$$

hence the claim. \square

Proposition 1.8. *Let $f : [a, b] \rightarrow \mathbb{R}_{\geq 0}$ be a continuous concave function, $f^\# : (0, b - a] \rightarrow \mathbb{R}_{\geq 0}$ its monotone reordering and $M := \max\{f(t) : t \in [a, b]\}$. If $M \geq b - a$ then $f^\#(t) \geq t$ for all $t \in (0, b - a]$.*

Proof. Since f is continuous, it achieves its maximum in some point $c \in [a, b]$. From now on suppose $c \in (a, b)$. If $c = a$ or $c = b$ the arguments are similar but easier. Set

$$g : [a, b] \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \frac{t-a}{c-a} \cdot (b-a) & \text{for all } a \leq t \leq c \\ \frac{b-t}{b-c} \cdot (b-a) & \text{for all } c \leq t \leq b. \end{cases}$$

Since length of $\{g \leq s\} = s$, we have $g^\#(t) = t$ for all $t \in (0, b - a]$. On the other hand, $g \leq f$ because $M \geq b - a$ and f is concave. Consequently,

$$g^\# \leq f^\#$$

by Prop. 1.4, which proves the claim. \square

2. THE ASYMPTOTIC VERSION OF DUMNICKI'S ALGORITHM

The asymptotic version of Dumnicki's reduction algorithm now reads as follows:

Theorem 2.1 (Asymptotic version of Dumnicki's reduction algorithm). *Let $m_1, \dots, m_{p-1}, m_p \in \mathbb{R}_{>0}$ and $P \subset \mathbb{R}_{\geq 0}^2$. For*

$$F : \mathbb{R}_{\geq 0}^2 \ni (\alpha_1, \alpha_2) \mapsto r_0 + r_1\alpha_1 + r_2\alpha_2, \quad r_0, r_1, r_2 \in \mathbb{R},$$

an affine function, define

$$\begin{aligned} P_1 &:= P \cap \{(\alpha_1, \alpha_2) : F(\alpha_1, \alpha_2) < 0\} \\ P_2 &:= P \cap \{(\alpha_1, \alpha_2) : F(\alpha_1, \alpha_2) > 0\}. \end{aligned}$$

If P_1 contains asymptotically (m_p) -non-special systems of dimension -1 and P_2 contains asymptotically (m_1, \dots, m_{p-1}) -non-special systems of dimension ≥ 0 , then P contains asymptotically (m_1, \dots, m_p) -non-special systems of dimension ≥ 0 .

Proof. By assumption, the set $n \cdot P_1 \cap \mathbb{N}_{\geq 0}^2$ contains an $m_p^{(n)}$ -non-special system of dimension -1 , and $n \cdot P_2 \cap \mathbb{N}_{\geq 0}^2$ contains an $(m_1^{(n)}, \dots, m_{p-1}^{(n)})$ -non-special system of dimension ≥ 0 , such that the $m_i^{(n)}$ satisfy the inequalities (ii) of Def. 0.4 for a given δ , for all $n \gg 0$.

Consequently, Dumnicki's reduction algorithm applied to $n \cdot P \cap \mathbb{N}_{\geq 0}^2$ and

$$F_n : \mathbb{R}_{\geq 0}^2 \ni (\alpha_1, \alpha_2) \mapsto nr_0 + r_1\alpha_1 + r_2\alpha_2,$$

shows that $n \cdot P \cap \mathbb{N}_{\geq 0}^2$ contains an $(m_1^{(n)}, \dots, m_p^{(n)})$ -non-special system of dimension ≥ 0 . Since the $m_i^{(n)}$ still satisfy the inequalities (ii) of Def. 0.4, for given δ , the algorithm is justified. \square

The facts shown in the last section can be used to prove a criterion for asymptotic (m) -non-specialty:

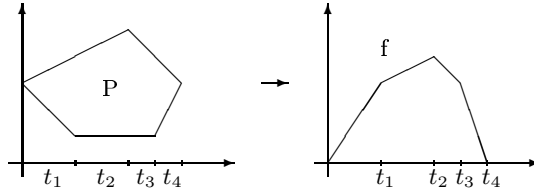
Theorem 2.2 (Criterion for asymptotic (m) -non-specialty). *Let P be a convex open subset of $\mathbb{R}_{\geq 0}^2$, such that its closure \overline{P} is compact. Set $[a, b] := p_x(\overline{P})$ where $p_x : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$ is the projection of $\mathbb{R}_{\geq 0}^2$ onto the positive x -axis. Define*

$$f : [a, b] \rightarrow \mathbb{R}_{\leq 0}, t \mapsto f(t) := \text{length of } p_x^{-1}(t) \cap \overline{P},$$

and let $f^\# : (0, b-a] \rightarrow \mathbb{R}_{\leq 0}$ be the monotone reordering of f .

If $m < b - a$ and $f^\#(t) \geq t$, for all $t \in (0, m]$, then P contains asymptotically (m) -non-special systems of dimension -1 .

The following diagram illustrates how we obtain the height function f from a convex polygon $P \subset \mathbb{R}_{\geq 0}^2$:



For the proof of the theorem we need some further properties of the function f :

Proposition 2.3. *Let P be a convex open subset of $\mathbb{R}_{\geq 0}^2$, \overline{P} compact. Define $f : [a, b] \rightarrow \mathbb{R}_{\leq 0}$ as in Thm. 2.2. Then f is concave and continuous on $[a, b]$.*

Proof. Let f^+ resp. f^- be the functions assigning to each $t \in [a, b]$ the upper resp. lower bound of the interval $p_x^{-1}(t) \cap P$. Then $f(t) = f^+(t) - f^-(t)$.

f^+ is concave: Choose $t_1, t_2 \in [a, b]$. Then the convexity of \overline{P} implies

$$(\lambda t_1 + (1 - \lambda)t_2, \lambda f^+(t_1) + (1 - \lambda)f^+(t_2)) \in \overline{P},$$

hence

$$\lambda f^+(t_1) + (1 - \lambda)f^+(t_2) \leq f^+(\lambda t_1 + (1 - \lambda)t_2).$$

Similarly f^- is convex, hence f as the difference of a concave and a convex function is concave.

The following lemma shows that f is also continuous on (a, b) and that $\lim_{t \rightarrow a} f^\pm(t)$ and $\lim_{t \rightarrow b} f^\pm(t)$ exist.

We still have to prove that $\lim_{t \rightarrow a} f^\pm(t) = f^\pm(a)$ resp. $\lim_{t \rightarrow b} f^\pm(t) = f^\pm(b)$: Closedness implies $(a, \lim_{t \rightarrow a} f^+(t)) \in \overline{P}$, hence $\lim_{t \rightarrow a} f^+(t) \leq f^+(a)$. If $\lim_{t \rightarrow a} f^+(t) < f^+(a)$ then

$$\lambda f^+(a) + (1 - \lambda) f^+(t) > (1 - \lambda) f^+(t)$$

for λ sufficiently close to 1, contradicting the concavity of f^+ . The same types of arguments hold for the other limits. \square

Lemma 2.4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a concave function. Then f is continuous on (a, b) , and the limits $\lim_{t \rightarrow a} f(t)$ and $\lim_{t \rightarrow b} f(t)$ exist.*

Proof. For any triple $a \leq t' < t_0 < t'' \leq b$, we have $t_0 = \frac{t'' - t_0}{t'' - t'} t' + \frac{t_0 - t'}{t'' - t'} t''$. The concavity of f implies

$$f(t_0) \geq \frac{t'' - t_0}{t'' - t'} f(t') + \frac{t_0 - t'}{t'' - t'} f(t'').$$

Subtracting $f(t')$ from both sides leads to the left hand inequality of

$$\frac{f(t_0) - f(t')}{t_0 - t'} \geq \frac{f(t'') - f(t')}{t'' - t'} \geq \frac{f(t'') - f(t_0)}{t'' - t_0},$$

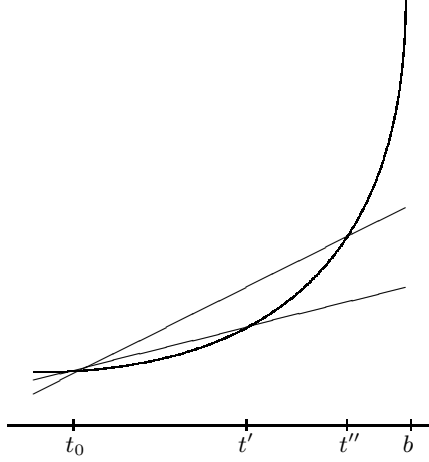
subtracting $f(t'')$ and multiplying with -1 to the right hand inequality. Renaming t_0, t', t'' the left hand inequality implies that the difference quotients $\frac{f(t'') - f(t_0)}{t'' - t_0}$ are increasing for $t'' \searrow t_0$, so they are bounded from below for t'' close to t_0 . Since $t_0 \in (a, b)$ there is always a $t' < t_0$ in (a, b) , hence the inequality chain shows that the difference quotients are also bounded from above. Consequently,

$$|f(t'') - f(t_0)| < M \cdot |t'' - t_0|$$

for appropriate $M > 0$ and $t'' > t_0$ close to t_0 . The same can be shown for $t' < t_0$ close to t_0 , hence f is continuous in t_0 .

The situation is more complicated for the boundary points a and b . If $\lim_{t \leq b} f(t)$ does not exist there are 3 possibilities:

- (1) There are at least 2 accumulation points, for different sequences $(t_n^\pm) \searrow b$, say $-\infty \leq y^- < y^+ \leq +\infty$. Then it is possible to choose a triple $t_0 < t_n^- < t_m^+$, $n, m \gg 0$, which contradicts the concavity of f (see the end of the proof in the proposition before).
- (2) $\lim_{t \leq b} f(t) = -\infty$: Then it is possible to choose a triple $t_0 < t < b$ contradicting the concavity of f as before.
- (3) $\lim_{t \leq b} f(t) = +\infty$: Then it is possible to choose $t_0 < t' < t'' < b$, such that the point $(t'', f(t''))$ lies over the line connecting $(t_0, f(t_0))$ and $(t', f(t'))$. But then the point $(t', f(t'))$ lies under the line segment connecting $(t_0, f(t_0))$ and $(t'', f(t''))$. This contradicts the concavity of f .



The same type of arguments hold for $\lim_{t \geq a} f(t)$. \square

Proof of Thm. 2.2. We start with a construction: Take any number $n \in \mathbb{N}$. Set

$$S_{(m_x, m_y)} := \left[\frac{m_x}{n} - \frac{1}{2n}, \frac{m_x}{n} + \frac{1}{2n} \right] \times \left[\frac{m_y}{n} - \frac{1}{2n}, \frac{m_y}{n} + \frac{1}{2n} \right], (m_x, m_y) \in \mathbb{N}^2.$$

The $S_{(m_x, m_y)}$ are closed squares of sidelength $\frac{1}{n}$ such that the coordinates of their centers are multiples of $\frac{1}{n}$. Then set $P_n := \bigcup_{S_{(m_x, m_y)} \subset \overline{P}} S_{(m_x, m_y)}$, the union of all such squares in \overline{P} .

Since $P_n \subset \overline{P}$ we have $p_x(P_n) \subset [a, b]$. Hence we can define the analog of f for P_n :

$$f_n : [a, b] \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \text{length of } p_x^{-1}(t) \cap P_n.$$

By construction, $f_n \leq f$. Furthermore the following holds:

Claim 1. For all $a < a_0 < b_0 < b$, the functions f_n converge uniformly against f on $[a_0, b_0]$, for $n \rightarrow \infty$.

Proof. Let f^+ and f^- be defined as in the proof of Prop. 2.3. This proof shows that f^+ and f^- are continuous functions, hence they are uniformly continuous on the compact interval $[a, b]$. Consequently, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f^\pm(x) - f^\pm(y)| < \epsilon \text{ for all } x, y \in [a, b].$$

Next, choose $a < a' < a_0 < b_0 < b' < b$. Since P is a convex set and the projection p_x is an open and continuous map, $p_x(P) = (a, b)$, and $p_x^{-1}(t) \cap P$ is a non-empty open interval, for all $t \in (a, b)$. Consequently, $f > 0$ on (a, b) , and f achieves a strictly positive minimum on $[a', b']$.

Let $\epsilon > 0$ be any real number such that 4ϵ is smaller than this minimum. Let $n \in \mathbb{N}$ be an integer such that $\frac{1}{n} < \epsilon$, $\frac{1}{n} < \delta$, $\frac{1}{n} < \min\{a_0 - a', b' - b_0, b_0 - a_0\}$. Let $\frac{k}{n} \in [a_0, b_0]$. By assumption,

$$f^+(k/n) - f^-(k/n) > 4\epsilon > \epsilon + \frac{1}{2n} + \frac{1}{n} + \frac{1}{2n} + \epsilon.$$

Hence the interval $[f^-(k/n) + \epsilon + \frac{1}{2n}, f^+(k/n) - \epsilon - \frac{1}{2n}]$ has length $> \frac{1}{n}$, consequently it contains at least one number of the form $\frac{m}{n}$.

Claim 1.1. $[\frac{k}{n} - \frac{1}{2n}, \frac{k}{n} + \frac{1}{2n}] \times [\frac{m}{n} - \frac{1}{2n}, \frac{m}{n} + \frac{1}{2n}] \subset \overline{P}$.

Proof. Let $t \in [\frac{k}{n} - \frac{1}{2n}, \frac{k}{n} + \frac{1}{2n}] \subset [a', b']$. Then $|t - \frac{k}{n}| < \delta$, hence $|f^\pm(t) - f^\pm(k/n)| < \epsilon$. This implies $f^+(t) > f^+(k/n) - \epsilon \geq \frac{m}{n} + \frac{1}{2n}$ and $\frac{m}{n} - \frac{1}{2n} \geq f^-(k/n) + \epsilon > f^-(t)$. \square

If $\frac{L}{n}$ is maximal among all $\frac{m}{n} \in [f^-(k/n) + \epsilon + \frac{1}{2n}, f^+(k/n) - \epsilon - \frac{1}{2n}]$, then $\frac{L}{n} \geq f^+(k/n) - \epsilon - \frac{1}{2n} - \frac{1}{n}$, and if $\frac{L}{n}$ is minimal, then $\frac{L}{n} \leq f^-(k/n) + \epsilon + \frac{1}{2n} + \frac{1}{n}$. Consequently

$$f^+(t) - (\frac{L}{n} + \frac{1}{2n}) < f^+(t) - (f^+(k/n) - \epsilon - \frac{1}{n}) \leq |f^+(t) - f^+(k/n)| + \epsilon + \frac{1}{n} \leq 2\epsilon + \frac{1}{n}.$$

Similarly we deduce $\frac{L}{n} - \frac{1}{2n} - f^-(t) < 2\epsilon + \frac{1}{n}$.

Now, $f(t) = f^+(t) - f^-(t) \geq f_n(t) > \frac{L}{n} - \frac{1}{n}$ for $t \in [\frac{k}{n} - \frac{1}{2n}, \frac{k}{n} + \frac{1}{2n}]$, hence

$$f(t) - f_n(t) \leq f^+(t) - \frac{L}{n} + \frac{1}{n} - f^-(t) < 4\epsilon + \frac{3}{n} < 7\epsilon.$$

Claim 1 is proven. \square

Claim 2. For every $\epsilon > 0$ and $n \gg 0$,

$$\|(f^\#)_{|(0,m]} - (f_n^\#)_{|(0,m]}\|_{\max} < \epsilon.$$

Proof. Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|(f^\#)_{|(0,b_0-a_0]} - (f_{|[a_0,b_0]})^\#\|_{\max} < \frac{\epsilon}{3}$$

as long as $(a_0 - a) + (b - b_0) < \delta$, by means of Thm. 1.6. Furthermore, there is an $n \gg 0$ such that $\|f - f_n\|_{\max} < \frac{\epsilon}{6}$, by Claim 1, hence also $\|f_{|[a_0,b_0]} - f_n|_{[a_0,b_0]}\|_{\max} < \frac{\epsilon}{6}$. Then Prop. 1.7 implies

$$\|(f_{|[a_0,b_0]})^\# - (f_n|_{[a_0,b_0]})^\#\|_{\max} < \frac{\epsilon}{3}.$$

Next we choose δ small enough to ensure $\|(f_n|_{[a_0,b_0]})^\# - (f_n^\#)_{|(0,b_0-a_0]}\|_{\max} < \frac{\epsilon}{3}$; this is possible again by Thm. 1.6.

Finally we choose a_0, b_0 such that in addition to $(a_0 - a) + (b - b_0) < \delta$, we also have $m < b_0 - a_0 < b - a$. By expanding $(f^\#)_{|(0,m]} - (f_n^\#)_{|(0,m]}$ to

$$(f^\#)_{|(0,m]} - (f_{|[a_0,b_0]})^\#_{|(0,m]} + (f_{|[a_0,b_0]})^\#_{|(0,m]} - (f_n|_{[a_0,b_0]})^\#_{|(0,m]} + (f_n|_{[a_0,b_0]})^\#_{|(0,m]} - (f_n^\#)_{|(0,m]}$$

we get the claim. \square

For all $\epsilon > 0$, Claim 2 implies $f_n^\#(t) > t - \epsilon$ for all $t \in [0, m]$ if $n \gg 0$.

Now for ϵ small enough, consider all integers e with $0 \leq e \leq \lfloor mn \rfloor - \lceil n\epsilon \rceil - 1$, and set $e' := \lfloor mn \rfloor - \lceil n\epsilon \rceil - e - 1$. Then $e + 1 + \lceil n\epsilon \rceil \leq \lfloor mn \rfloor$, hence $\frac{e+1+\lceil n\epsilon \rceil}{n} \leq m$, and we can apply $f_n^\#$ on $\frac{e+1+\lceil n\epsilon \rceil}{n}$:

$$f_n^\#(\frac{e+1+\lceil n\epsilon \rceil}{n}) = f_n^\#(\frac{\lfloor mn \rfloor - e'}{n}) > \frac{\lfloor mn \rfloor - e'}{n} - \epsilon \geq \frac{\lfloor mn \rfloor - e' - \lceil n\epsilon \rceil}{n} = \frac{e+1}{n},$$

for $n \gg 0$. Consequently for each such e the step function $f_n^\#$ has a step of height at least $\frac{e+1}{n}$, and these steps can be chosen pairwise distinct.

On the other hand, $f_n^\#$ is just a reordering of the steps in f_n , so f_n has the same property. By construction the height of the step of f_n over $\frac{k}{n}$ counts the number of points $(\frac{k}{n}, \frac{l}{n})$ inside P . But this means that $n \cdot P \cap \mathbb{N}_{\geq 0}^2$ contains a non-special linear

system $\mathcal{L}_{D_n}(\lfloor mn \rfloor - \lceil n\epsilon \rceil)$ of dimension -1 , by Dumnicki's non-specialty criterion. Since

$$\left| \frac{\lfloor mn \rfloor - \lceil n\epsilon \rceil - nm}{nm} \right| \leq \left| \frac{\lfloor mn \rfloor - nm}{nm} \right| + \frac{\lceil n\epsilon \rceil}{nm} \leq \frac{1}{mn} + \frac{\epsilon}{m} + \frac{1}{nm} \rightarrow 0$$

for $\epsilon \rightarrow 0$, $n \rightarrow \infty$, the theorem is proven. \square

3. A LOWER BOUND FOR THE SESHADRI CONSTANT OF 10 POINTS IN \mathbb{CP}^2

The following proposition allows to prove the nefness criterion Prop. 0.5:

Proposition 3.1. *Let $m_1, \dots, m_r \in \mathbb{N}_{>0}$, let $D_1 \subset D_2 \subset \mathbb{N}^2$ be finite subsets, and let $p_1, \dots, p_r \in \mathbb{P}^2$ be points. If $\mathcal{L}_{D_1}(m_1 p_1, \dots, m_r p_r)$ has expected dimension ≥ 0 , then also $\mathcal{L}_{D_2}(m_1 p_1, \dots, m_r p_r)$.*

Proof. The subspace $\mathcal{L}_{D_1}(m_1 p_1, \dots, m_r p_r) \subset \mathcal{L}_{D_2}(m_1 p_1, \dots, m_r p_r)$ is described by the intersection of $\mathcal{L}_{D_2}(m_1 p_1, \dots, m_r p_r) \subset \mathbb{P}(\sum_{i+j \leq d} a_{ij} x^i y^j)$ with the linear subspace

$$\{a_{ij} = 0 : (i, j) \in D_2 \setminus D_1\}.$$

Here, d is chosen such that $D_1 \subset D_2 \subset \{(i, j) : i + j \leq d\}$. Consequently,

$$\dim \mathcal{L}_{D_1}(m_1 p_1, \dots, m_r p_r) \geq \dim \mathcal{L}_{D_2}(m_1 p_1, \dots, m_r p_r) - \#(D_2 \setminus D_1).$$

In particular, $\mathcal{L}_{D_1}(m_1 p_1, \dots, m_r p_r)$ cannot have expected dimension if $\mathcal{L}_{D_2}(m_1 p_1, \dots, m_r p_r)$ has not. \square

Proof of Prop. 0.5. The assumption on P implies that there exists a sequence $\delta_n \xrightarrow{>} 0$, natural numbers $d_n, m_1^{(n)}, \dots, m_r^{(n)}$ for all $n \in \mathbb{N}$, $d_n \rightarrow \infty$ for $n \rightarrow \infty$, and subsets $D_n \subset d_n \cdot P \cap \mathbb{N}_{\geq 0}^2$ such that $\mathcal{L}_{D_n}(m_1^{(n)}, \dots, m_r^{(n)})$ is non-special of dimension ≥ 0 and

$$\left| \frac{m_i^{(n)}}{d_n \cdot m} - 1 \right| = \left| \frac{m_i^{(n)} - d_n m}{d_n m} \right| < \delta_n, \quad i = 1, \dots, r.$$

Let $m_n := \min\{m_1^{(n)}, \dots, m_r^{(n)}\}$. Then Prop. 3.1 implies that $\mathcal{L}_{d_n}(m_n^r)$ is non-special of dimension ≥ 0 .

Since $\frac{d_n}{m_n - 1}$ and $\frac{d_n}{m_n}$ have the same limit, and for some $i_n \in \{1, \dots, r\}$,

$$\frac{d_n}{m_n} = \frac{d_n}{m_{i_n}^{(n)}} = \frac{d_n m}{m_{i_n}^{(n)}} \cdot \frac{1}{m} \rightarrow \frac{1}{m},$$

the proposition follows. \square

Proof of Thm. 0.6. In the following diagram, let the points O, A, B, \dots, R, S be given by the coordinates

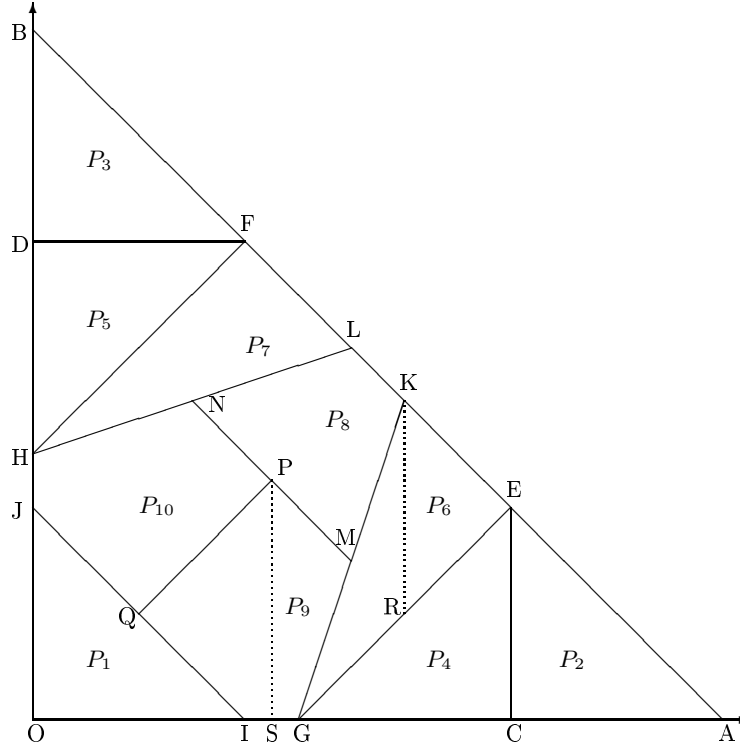
$$O = (0, 0), \quad A = (1, 0), \quad C = \left(\frac{9}{13}, 0\right), \quad E = \left(\frac{9}{13}, \frac{4}{13}\right), \quad G = \left(\frac{5}{13}, 0\right),$$

$$B = (0, 1), \quad D = \left(0, \frac{9}{13}\right), \quad F = \left(\frac{4}{13}, \frac{9}{13}\right), \quad H = \left(0, \frac{5}{13}\right),$$

$$I = \left(\frac{4}{13}, 0\right), \quad K = \left(\frac{7}{13}, \frac{6}{13}\right), \quad M = \left(\frac{6}{13}, \frac{3}{13}\right),$$

$$J = \left(0, \frac{4}{13}\right), \quad L = \left(\frac{6}{13}, \frac{7}{13}\right), \quad N = \left(\frac{3}{13}, \frac{6}{13}\right),$$

$$P = \left(\frac{9}{26}, \frac{9}{26}\right), \quad Q = \left(\frac{2}{13}, \frac{2}{13}\right), \quad R = \left(\frac{7}{13}, \frac{2}{13}\right), \quad S = \left(\frac{9}{26}, 0\right).$$



The diagram is possible since E, K, L, F lie on the line AB , the points I, G, C on the line OA , the points J, H, D on the line OB , the point M on the line GK , the point N on the line HL , the point P on the line NM and Q on the line IJ . Furthermore, S lies between I and G on OA , and R lies on the line GE .

The diagram shows the dissection of the 2-dimensional simplex OAB into 10 polygons P_1, \dots, P_{10} by straight lines. The indices of the polygons denote the sequence of dissections. To prove the theorem we apply the asymptotic version of Dumnicki's reduction algorithm to this sequence of dissections. That is, we have to show that for every $m < \frac{4}{13}$ each of the polygons P_i , $i = 1, \dots, 9$, contains asymptotically (m) -non-special systems of dimension -1 , and that P_{10} contains asymptotically (m) -non-special systems of dimension ≥ 0 .

By construction the polygons are convex. Hence Thm. 2.2 together with Prop. 1.8 and Prop. 2.3 imply that it is enough to show the following, for every polygon P_i : The projection of P_i onto the x -axis is an interval of length $\geq \frac{4}{13} > m$, and there is a vertical section of P_i of length $\geq \frac{4}{13} > m$. By symmetry, it is also possible to show these inequalities for the projection onto the y -axis and a horizontal section. Furthermore, since the lengths are $> m$, it will be always possible to add some monomials to $D_{10}^{(n)}$ in $P_{10}^{(n)}$. Prop. 3.1 shows that this produces $m_{10}^{(n)}$ -non-special systems of dimension ≥ 0 in $P_{10}^{(n)}$, for $n \gg 0$.

For the polygons P_1, \dots, P_5 the inequalities are obvious. For the polygon P_6 the projection to the x -axis is the interval GC which has length $\frac{4}{13}$. The vertical section KR has also length $\frac{4}{13}$. By symmetry, P_7 also satisfies the inequalities.

The projection of P_8 onto the x -axis is the interval $[\frac{3}{13}, \frac{7}{13}]$ (these are the x -coordinates of N and M), and the vertical section LM has length $\frac{4}{13}$. The projection of P_9 onto the x -axis is $[\frac{2}{13}, \frac{6}{13}]$, and the vertical section PS has length $\frac{4}{13}$. By symmetry, P_{10} also satisfies the necessary inequalities. \square

Remark 3.2. Since the bound $\frac{13}{4}$ is rational there might be a pair (d, m) with $\mathcal{L}_d(10^{m+1})$ non-special of non-negative dimension and $\frac{d}{m} = \frac{13}{4}$. From such a pair the theorem would follow by Thm. 0.1 without any limit process. But it is difficult to find such a pair: $\mathcal{L}_d(10^{m+1})$ has expected dimension -1 up to $m = 92$, and then it is still not clear how to prove non-specialty. For example, the cutting proposed in the proof of the theorem above might require an even bigger m .

REFERENCES

- [Bir99] Paul Biran. Constructing new ample divisors out of old ones. *Duke Math. J.*, 98(1):113–135, 1999.
- [CM01] C. Ciliberto and R. Miranda. The Segre and Harbourne–Hirschowitz conjectures. In *Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001)*, volume 36 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 37–51. Kluwer Acad. Publ., Dordrecht, 2001.
- [DJ05] M. Dumnicki and W. Jarnicki. New effective bounds on the dimension of a linear system in \mathbb{P}^2 . arXiv:math/0505183, 2005.
- [Dum06] M. Dumnicki. Reduction method for linear systems of plane curves with base fat points. arXiv:math/0606716, 2006.
- [Eck05] Thomas Eckl. Seshadri constants via lelong numbers. preprint math.AG/0508561, to be published in *Math. Nachr.*, August 2005.
- [Har03] Brian Harbourne. Seshadri constants and very ample divisors on algebraic surfaces. *J. Reine Angew. Math.*, 559:115–122, 2003.
- [HR03] B. Harbourne and J. Roé. Computing multi-point Seshadri constants on \mathbb{P}^2 . preprint, arXiv:math/0309064v3, 2003.
- [HR04] Brian Harbourne and Joaquim Roé. Linear systems with multiple base points in \mathbb{P}^2 . *Adv. Geom.*, 4(1):41–59, 2004.
- [HR05] B. Harbourne and J. Roé. Multipoint Seshadri constants on \mathbb{P}^2 . *Rend. Sem. Mat. Univ. Politec. Torino*, 63(1):99–102, 2005.
- [Mir99] R. Miranda. Linear Systems of plane curves. *Notices AMS*, 46:192–201, 1999.
- [Nag59] M. Nagata. On the 14-th problem of Hilbert. *Amer. J. Math.*, 81:766–772, 1959.
- [Sch07] F. Schüller. Ein neuer ansatz zur harbourne-hirschowitz-vermutung. diploma thesis, Universität zu Köln, 2007. <http://www.mi.uni-koeln.de/~kebekus/teaching/diplomarbeiten.html>.
- [STG02] Tomasz Szemberg and Halszka Tutaj-Gasińska. General blow-ups of the projective plane. *Proc. Amer. Math. Soc.*, 130(9):2515–2524 (electronic), 2002.
- [TG03] Halszka Tutaj-Gasińska. A bound for Seshadri constants on \mathbb{P}^2 . *Math. Nachr.*, 257:108–116, 2003.
- [Xu94] Geng Xu. Curves in \mathbf{P}^2 and symplectic packings. *Math. Ann.*, 299(4):609–613, 1994.

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